L_∞ algebra structures of Lie algebra deformations

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Abstract

In this paper,we will show how to kill the obstructions to Lie algebra deformations via a method which essentially embeds a Lie algebra into Strong homotopy Lie algebra or L_∞ algebra. All such obstructions have been transfered to the revelvant L_∞ algebras which contain only three terms.

1 Introduction

In the last two decades, deformations of various types of structures have assumed an ever increasing role in mathematics and physics. For each such deformation problem a goal is to determine if all related deformation obstructions vanish and many beautiful techniques been developed to determine when this is so. Sometimes genuine deformation obstructions arise and occasionally that closes mathematical development in such cases, but in physics such problems are dealt with by introducing new auxiliary fields to kill such obstructions. This idea suggests that one might deal with deformation problems by enlarging the relevant category to a new category obtained by appending additional algebraic structures to the old category. To achieve the purpose of removing obstructions to Lie algebra deformation, we embed the Lie algebra into an appropriate sh-Lie algebra in such a way that the obstructions will vanish in the category of sh-Lie algebra deformations. In order to be complete we review basic facts on Lie algebra deformations; more detail may be found in the book edited by M.Hazawinkel and M.Gerstenhaber [4].

2 Deformation theory, sh-Lie algebras

Let A be a k-algebra and α be its multiplication, i.e., α is a k-bilinear map $A \times A \longrightarrow A$ defined by $\alpha(a, b) = ab$. A deformation of A may be defined to be a formal power series $\alpha_t = \alpha + t\alpha_1 + t^2\alpha_2 + \cdots$ where each $\alpha_i : A \times A \longrightarrow A$ is a k-bilinear map and the "multiplication" α_t is formally of the same "kind" as α, e.g., it is associative or Lie or whatever is required. One technique used to set up a deformation problem is to extend a k-bilinear mapping $\alpha_t : A \times A \longrightarrow A[[t]]$ to a $k[[t]]$-bilinear mapping $\alpha_t : A[[t]] \times A[[t]] \longrightarrow A[[t]]$. A mapping $\alpha_t : A[[t]] \times A[[t]] \longrightarrow A[[t]]$ obtained in this manner is necessarily uniquely determined by it's values on $A \times A$. In fact we would not regard the mapping $\alpha_t : A[[t]] \times A[[t]] \longrightarrow A[[t]]$ to be a deformation of A unless it is determined by it's values on $A \times A$.

From this point on, we assume that (A, α) is a Lie algebra,i.e., we assume that $\alpha(\alpha(a, b), c) + \alpha(\alpha(b, c), a) + \alpha(\alpha(c, a), b) = 0$. Thus the problem of deforming a Lie algebra A is equivalent to the problem of finding a mapping $\alpha_t : A \times A \longrightarrow A[[t]]$ such that $\alpha_t(\alpha_t(a, b), c) + \alpha_t(\alpha_t(b, c), a) + \alpha_t(\alpha_t(c, a), b) = 0$. If we set $\alpha_0 = \alpha$ and expand this Jacobi identity by making the substitution

$\alpha_t = \alpha + t\alpha_1 + t^2\alpha_2 + \cdots$, we get the equation

$$\sum_{i,j=0}^{\infty} [\alpha_j(\alpha_i(a,b),c) + \alpha_j(\alpha_i(b,c),a) + \alpha_t(\alpha_i(c,a),b)]t^{i+j} = 0 \tag{1}$$

and consequently a sequence of deformation equations;

$$\sum_{i,j\geq 0, i+j=n} [\alpha_j(\alpha_i(a,b),c) + \alpha_j(\alpha_i(b,c),a) + \alpha_t(\alpha_i(c,a),b)] = 0. \tag{2}$$

The first two equations are:

$$\alpha_0(\alpha_0(a,b),c) + \alpha_0(\alpha_0(b,c),a) + \alpha_0(\alpha_0(c,a),b) = 0 \tag{3}$$

$$\begin{aligned} \alpha_0(\alpha_1(a,b),c) + \alpha_0(\alpha_1(b,c),a) + \alpha_0(\alpha_1(c,a),b) + \alpha_1(\alpha_0(a,b),c) \\ +\alpha_1(\alpha_0(b,c),a) + \alpha_1(\alpha_0(c,a),b) = 0 \end{aligned} \tag{4}$$

We can reformulate the discussion above in a slightly more compact form. Given a sequence $\alpha_n : A \times A \longrightarrow A$ of bilinear maps, we define "compositions" of various of the α_n as follows:

$$\alpha_i\alpha_j : A \times A \times A \longrightarrow A \tag{5}$$

is defined by

$$(\alpha_i\alpha_j)(x_1,x_2,x_3) = \sum_{\sigma\in unsh(2,1)} (-1)^\sigma \alpha_i(\alpha_j(x_{\sigma(1)},x_{\sigma(2)}),x_{\sigma(3)}) \tag{6}$$

for arbitrary $x_1, x_2, x_3 \in A$.

Thus the deformation equations are equivalent to following equations:

$$\begin{aligned} \alpha_0^2 &= 0 && (7)\\ \alpha_0\alpha_1 + \alpha_1\alpha_0 &= 0 && (8)\\ \alpha_1^2 + \alpha_0\alpha_2 + \alpha_2\alpha_0 &= 0 && (9)\\ &\cdots \\ \Sigma_{i+j=n}\alpha_i\alpha_j &= 0 && (10)\\ &\cdots. && (11) \end{aligned}$$

Define a bracket on the sequence $\{\alpha_n\}$ of mappings by $[\alpha_i, \alpha_j] = \alpha_i\alpha_j + \alpha_j\alpha_i$ and a "differential" d by $d = ad_{\alpha_0} = [\alpha_0, \cdot]$, the "adjoint representation" relative to α_0. Notice that the second equation in the list above is equivalent

to the statement that α_1 defines a cocycle $\alpha_1 \in Z^2(A, A)$ in the Lie algebra cohomology of A. Moreover it is known that the second cohomology group $H^2(A, A)$ classifies the equivalence class of infinitesimal deformations of A [4]. This being the case we refer to the triple (A, α_0, α_1) as being initial conditions for deforming the Lie algebra A. Notice that the third equation in the above list can be rewritten as

$$[\alpha_1, \alpha_1] = -[\alpha_0, \alpha_2] = -d\alpha_2 \tag{12}$$

When this equation holds one has then that $[\alpha_1, \alpha_1]$ is a coboundary and so defines the trivial element of $H^3(A, A)$ for any given deformation α_t. Thus if $[\alpha_1, \alpha_1]$ is not a coboundary, then we may regard $[\alpha_1, \alpha_1]$ as the first obstruction to deformation and in this case we can not deform A at second order. In general, to say that there exists a deformation of (A, α_0, α_1) up to order $n-1$, means that there exists a sequence of maps $\alpha_0, \cdots, \alpha_{n-1}$ such that $\sum_{\sigma \in unsh(2,1)} (-1)^\sigma \alpha_t(\alpha_t(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)}) = 0 \quad (mod \quad t^n)$. If this is the case and if there is an obstruction to deformation at nth order, then it follows that $\rho_n = -(\Sigma_{i+j=n, i,j>0} \alpha_i \alpha_j)$ is in some sense the obstruction and $[\rho_n]$ is a nontrivial element of $H^3(A, A)$. If $[\rho_n] \neq 0$,then the process of obtaining a deformation will terminate at order $n-1$ due to the existence of the obstruction ρ_n. In principal, it is possible that one could return to the beginning and select different terms for the α_i but when this fails what can one say? This is the issue in the remainder of this section.

Indeed the central point of this section is to show that when there is an obstruction to the deformation of a Lie algebra, one can use the obstruction itself to define one of the structure mappings of an sh-Lie algebra. Without loss of generality, we consider a deformation problem which has a first order obstruction.

The required sh-Lie structure lives on a graded vector space X_* which we define below. This space in degree zero is given by $X_0 = A[[t]] = (\{\Sigma a_i t^i | \quad a_i \in A\}$. The spaces $\mathcal{B} = < t^2 >= A[[t]] \cdot t^2 = \{\Sigma_{i \geq 2} a_i t^i | a_i \in A\}$ and $\mathcal{F} = X_0/\mathcal{B} =$ are also relevant to our construction. Notice that $\mathcal{F}$ is isomorphic to $\{a_0 + a_1 t| \quad a_0, a_1 \in A\}$ as a linear space and that $X_0, \mathcal{B}$ are both $k[[t]]$-modules while $\mathcal{F}$ is a $k[[t]]/ < t^2 >$ module (recall that k is underlying field of A). To summarize, we have following short exact sequence:

$$0 \longrightarrow \mathcal{B} \longrightarrow X_0 \longrightarrow \mathcal{F} \longrightarrow 0.$$

Suppose that the initial Lie structure of A is given by $\alpha_0 : A \times A \longrightarrow A$ and denote a fixed infinitesimal deformation by $[\alpha_1] \in H^2(A, A)$. One of

the structure mappings of our sh-Lie structure will be determined by the mapping $\tilde{l}_2 : X_0 \times X_0 \longrightarrow X_0$ defined as follows: for any $a, b \in A$, let

$$\tilde{l}_2(a, b) = \alpha_0(a, b) + \alpha_1(a, b)t \tag{13}$$

and extend it to X_0 by requiring that it be $k[[t]]$-bilinear. Obviously, $\tilde{l}_2$ induces a Lie bracket $[,]$ on $\mathcal{F}$, but if the obstruction $[\alpha_1, \alpha_1]$ is not zero, then $\tilde{l}_2^2 \neq 0$ and consequently $\tilde{l}_2$ can not be a Lie bracket on X_0 (since it doesn't satisfy the Jacobi identity).

To deal with this obstruction we will show that we can use α_0, α_1 to construct an sh-Lie structure with at most three nontrivial structure maps l_1, l_2, l_3 such that the value of l_3 on $A \times A \times A$ is the same as that of $[\alpha_1, \alpha_1]$. In particular, l_3 will vanish if and only if the obstruction $[\alpha_1, \alpha_1]$ vanishes. Thus the sh-Lie algebra encodes the obstruction to deformation of the Lie algebra (A, α_0).

The required sh-Lie algebra lives on a certain homological resolution (X_*, l_1) of $\mathcal{F}$, so our first task is to construct this resolution space for $\mathcal{F}$. To do this let's introduce a "superpartner set of A," denoted by $A[1]$, as follows: for each $a \in A$, introduce a^* such that $a^* \leftrightarrow a$ is a one to one correspondence and define $\epsilon(a^*) = \epsilon(a) + 1$. Let $X_1 = A[1][[t]]t^2$ and define a map $l_1 : X_1 \longrightarrow X_0$ by

$$l_1(x) = \Sigma_{i \geq 2} a_i t^i \in X_0, \qquad x = \Sigma_{i \geq 2} a_i^* t^i \in X_1.$$

Notice that this is just the $k[[t]]$ extension of the $a^* \leftrightarrow a$ map. Since l_1 is injective, we obtain a homological resolution $X_* = X_0 \oplus X_1$ due to the fact that the complex defined by:

$$0 \longrightarrow X_1 \xrightarrow{l_1} X_0 \longrightarrow 0 \tag{14}$$

has the obvious property that $H(X_*) = H_0(X_*) \simeq \mathcal{F}$.

The *sh*-Lie algebra being constructed will have the property that $l_n = 0, n \geq 4$. Generally sh-Lie algebras can have any number of nontrivial structure maps. The fact that all the structure mappings of our sh-Lie algebra are zero with the exception of l_1, l_2, l_3 is an immediate consequece of the fact that we are able to produce a resolution of the space $\mathcal{F}$ such that $X_k = 0$ for $k \geq 2$. In general such resolutions do not exist and so one does not have $l_n = 0$ for $n \geq 4$.

In order to finish the preliminaries, we now construct a contracting homotopy s such that following commutative diagram holds:

$$\begin{array}{ccccccc}
0 \longrightarrow & X_1 & \overset{s}{\underset{l_1}{\overset{\longleftarrow}{\longrightarrow}}} & X_0 & \longrightarrow 0 \\
& \lambda\Big\uparrow\Big\downarrow\eta & & \lambda\Big\uparrow\Big\downarrow\eta & \\
0 \longrightarrow & 0 & \longrightarrow & \mathcal{F} & \longrightarrow 0
\end{array}$$

Clearly the linear space X_0 is the direct sum of $\mathcal{B}$ and a complementary subspace which is isomorphic to $\mathcal{F}$; consequently we have $X_0 \simeq \mathcal{B} \oplus \mathcal{F}$. Define $\eta = proj\ |_{\mathcal{F}}, \lambda = i_{\mathcal{F}\to X_0}$ and a contracting homotopy $s : X_0 \longrightarrow X_1$ as follows: write $X_0 = \mathcal{B} \oplus \mathcal{F}$, set $s|_{\mathcal{F}} = 0$, and let $s(x) = -x^*$ for all $x \in \mathcal{B}$. It is easy to show that $\lambda \circ \eta - 1_{X_*} = l_1 \circ s + s \circ l_1$. In order to obtain the sh-Lie algebra referred to above, we apply a theorem of [2]. The hypothesis of this theorem requires the existence of a bilinear mapping $\tilde{l}_2$ from $X_0 \times X_0$ to X_0 with the properties that for $c, c_1, c_2, c_3 \in X_0$ and $b \in \mathcal{B}$ (i) $\tilde{l}_2(c, b) \in \mathcal{B}$ and (ii) $\tilde{l}_2^2(c_1, c_2, c_3) \in \mathcal{B}$. To see that (i) holds notice that if $p(t), q(t) \in X_0 = A[[t]]$, then $\tilde{l}_2(p(t), q(t)t^2) = r(t)t^2$ for some $r(t) \in A[[t]] = X_0$. Also note that the fact that $\tilde{l}_2$ induces a Lie bracket on $\mathcal{F} = X_0/\mathcal{B}$ implies that $\tilde{l}_2^2$ is zero modulo $\mathcal{B}$ and (ii) follows. Thus X_* supports an sh-Lie structure with only three nonzero structure maps l_1, l_2, l_3 (see the remark at the end of [2]).

Theorem 1 *Given a Lie algebra A with Lie bracket α_0 and an infinitesimal obstruction $[\alpha_1] \in H^2(A, A)$ to deforming (A, α_0), there is an sh-Lie algebra on the graded space (X_*, l_1) with structure maps $\{l_i\}$ such that $l_n = 0$ for $n \geq 4$. The graded space X_* has at most two nonzero terms $X_0 = A[[t]], X_1 = A[1][[t]]t^2$. Finally, the maps l_1, l_2, l_3 may be given explicitly in terms of the maps α_0, α_1.*

Remark : The mapping l_1 is simply the differential of the graded space (X_*, l_1). The mapping l_2 restricted to $X_0 \times X_0$ is the mapping $\tilde{l}_2$ defined directly in terms of α_0, α_1 above. On $X_1 \times X_0$, l_2 is determined by $l_2(a^*t^2, b) = t^2(\alpha_0(a, b)^* + \alpha_1(a, b)^*t)$ for $a^* \in A[1], b \in A$. Finally, l_3 is uniquely determined by its values on $A \times A \times A \subset X_0 \times X_0 \times X_0$ and is explicitly a multiple of the obstruction to the deformation of (A, α_0), in particular, $l_3(a_1, a_2, a_3) = -t^2[\alpha_1, \alpha_1](a_1, a_2, a_3), a_i \in A$.

Proof. The sh-Lie structure maps are given by Theorem 7 of [2]. The fact that $l_n = 0, n \geq 4$ is an observation of Markl which was proved by Barnich [**?**] (see the remark at the end of [2]). A generalization of Markl's remark is available in a paper by Al-Ashhab [1] and in that paper more explicit formulas are given for l_1, l_2, l_3. Examination of these formulas provide the details needed for the calculations below.

First of all, we examine the mapping $l_2 : X_* \times X_* \longrightarrow X_*$. Now $l_2 : A \times A \longrightarrow X_*$ is determined by $\tilde{l}_2 : X_0 \times X_0 \longrightarrow X_0$, consequently we need only consider the restricted mapping:

$$l_2 : X_1 \times X_0 \longrightarrow X_1. \tag{15}$$

Moreover, since X_0 is a module over $k[[t]]$, X_1 is a module over $k[[t]]t^2$, and $\tilde{l}_2$ respects these structures we need only consider its values on pairs (a^*t^2, b) with $a^*t^2 \in X_1, b \in X_0$. By Theorem 2.2 of [1], we have

$$\begin{aligned} l_2(a^*t^2, b) &= -sl_2l_1[(a^*t^2) \otimes b] \\ &= -sl_2[l_1(a^*t^2) \otimes b + (-1)^{\epsilon(a^*)}(a^*t^2) \otimes l_1(b)] \\ &= -sl_2[(at^2 \otimes b)] = -s[t^2 l_2(a \otimes b)] \\ &= -s[t^2(\alpha_0(a,b) + \alpha_1(a,b)t)] \\ &= -s[\alpha_0(a,b)t^2 + \alpha_1(a,b)t^3] \\ &= \alpha_0(a,b)^*t^2 + \alpha_1(a,b)^*t^3 \\ &= t^2(\alpha_0(a,b)^* + \alpha_1(a,b)^*t). \end{aligned} \tag{16}$$

From this deduction, we that the mapping l_2 can essentially be replaced by the modified map:

$$\bar{l}_2 : A[1] \times A \longrightarrow A[1][[t]], \qquad \bar{l}_2(a^*, b) = \alpha_0(a,b)^* + \alpha_1(a,b)^*t. \tag{17}$$

We clarify this remark below by showing that a new sh-Lie structure can be obtained with $\bar{l}_2$ playing the role of l_2.

The next mapping we examine is the mapping

$$l_3 : X_0 \times X_0 \times X_0 \longrightarrow X_1 \tag{18}$$

Since l_3 is $k[[t]]$-linear,we need only consider mappings of the type:
$l_3 : A \times A \times A \longrightarrow X_1$ where for $x_1, x_2, x_3 \in A$,

$$l_3(x_1, x_2, x_3) = sl_2^2(x_1, x_2, x_3)$$

$$
\begin{aligned}
&= \sum_{\sigma\in unsh(2,1)} (-1)^{\sigma} s l_2(l_2(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)}) \\
&= \sum_{\sigma\in unsh(2,1)} (-1)^{\sigma} s l_2(\alpha_0(x_{\sigma(1)}, x_{\sigma(2)}) + \alpha_1(x_{\sigma(1)}, x_{\sigma(2)})t, x_{\sigma(3)}) \\
&= \sum_{\sigma\in unsh(2,1)} (-1)^{\sigma} s[\alpha_0(\alpha_0(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)}) + \\
&\quad t\alpha_1(\alpha_0(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)}) + t\alpha_0(\alpha_1(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)}) \\
&\quad + t^2\alpha_1(\alpha_1(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)}), x_{\sigma(3)}) \\
&= s(\sum_{\sigma\in unsh(2,1)} (-1)^{\sigma} \alpha_0(\alpha_0(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)}) + \\
&\quad + t(\sum_{\sigma\in unsh(2,1)} (-1)^{\sigma} \alpha_1(\alpha_0(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)}) + \\
&\quad \sum_{\sigma\in unsh(2,1)} (-1)^{\sigma} \alpha_0(\alpha_1(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)})) \\
&\quad + t^2(\sum_{\sigma\in unsh(2,1)} (-1)^{\sigma} \alpha_1(\alpha_1(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)}))] \\
&= s((\alpha_0^2 + t(\alpha_0\alpha_1 + \alpha_1\alpha_0) + t^2\alpha_1^2)(x_1, x_2, x_3)) \\
&= s(t^2\alpha_1^2(x_1, x_2, x_3)) \\
&= -t^2(\alpha_1^2(x_1, x_2, x_3))^* \qquad (19)
\end{aligned}
$$

Or $l_3(x_1, x_2, x_3) = -t^2([\alpha_1, \alpha_1](x_1, x_2, x_3))^*$ which is precisely the "first deformation obstruction class".

Recall that we know from Theorem 7 of [2] that we have an sh-Lie structure. The point of these calculations is that it enables us to obtain the modified sh-Lie structure of Corollary 10 below and it is this structure which is relevant to Lie algebra deformation. Thus we already know that the mappings l_1, l_2, l_3 satisfy the relations:

$$l_1 l_2 - l_1 l_2 = 0 \qquad (20)$$

$$l_2{}^2 + l_1 l_3 + l_3 l_1 = 0 \qquad (21)$$

$$l_3{}^2 = 0 \qquad (22)$$

$$l_2 l_3 + l_3 l_2 = 0. \qquad (23)$$

Observe that if we let $\tilde{X}_* = \tilde{X}_1 \oplus \tilde{X}_0 = A[1][[t]] \oplus A[[t]]$, then the formulas defining l_1, l_2, l_3 defined on X_* make sense on the new complex $\tilde{X}_*$. Indeed the calculations above show that l_1, l_3 are uniquely determined by their values on "constants" in the sense that they could be first defined on elements of

$A[1] \oplus A \subseteq A[1][[t]] \oplus A[[t]]$ and then extended to $A[1][[t]] \oplus A[[t]]$ using the fact that l_1, l_3 are required to be $k[[t]]$ linear. l_2 is not obviously $k[[t]]$ linear. The whole point of corollary 10 below is that the *sh*-Lie structure defined by Theorem 10 can be redefined to obtain *sh*-Lie maps on the graded space $\tilde{X}_*$ which are obviously $k[[t]]$ linear and consequently this "new" structure is intimately related to deformation theory. Thus, as we say above, the modified map $\bar{l}_2$ can be extended to the new complex $\tilde{X}_*$ and is uniquely determined by its values on "constants". If we denote the extensions of l_1, l_3 to $\tilde{X}_*$ by $\bar{l}_1, \bar{l}_3$, then clearly these mappings satisfy the same relations (63)-(66) as the maps l_1, l_2, l_3 and consequently if we define $\bar{l}_n = 0, n \geq 4$ it follows that $(\tilde{X}_*, \tilde{l}_1, \tilde{l}_2, \tilde{l}_3, 0, 0 \cdots)$ is an *sh*-Lie algebra. This proves the following corollary.

Corollary 2 *There is an sh-Lie structure on* $A[1][[t]] \oplus A[[t]]$ *whose structure mappings* $\{\bar{l}_1, \bar{l}_2, \bar{l}_3, 0, \cdots\}$ *are precisely the mappings* $\{l_1, l_2, l_3, 0, \cdots\}$ *when restricted to* $A[1][[t]]t^2 \oplus A[[t]]$*. Moreover, the structure mappings of* $A[1][[t]] \oplus A[[t]]$ *have the property that they are uniquely determined by their values on* $A[1] \oplus A$ *and* $k[[t]]$ *linearity.*

From the discussion above the set of mappings $\{\bar{l}_1, \bar{l}_2, \bar{l}_3\}$ is essentially a deformation of an *sh*-Lie algebra. In addition, the construction of the mapping $\bar{l}_2$ is equivalent to defining an initial condition for a Lie algebra deformation.

This means that a Lie algebra which can't be deformed in the category of Lie algebra may admit an *sh*-Lie algebra deformation by first imbedding it into an appropriate *sh*-Lie algebra.

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